Existence of n-th root

PROVING THE EXISTENCE OF N-TH ROOT is the topic of this note.

This note is an experiment in including PDF pages. The included PDF pages are scans of handwriting with a fountain pen. The proof is a modification of the proof of the existence of the square root from Chapter 2 of the *Lectures on Real Analysis* textbook[1] to the n-th root case.

[1] F. Lárusson. *Lectures on Real Analysis*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2012. ISBN 9781107026780. URL https://books.google.com/books?id=koj-IrXXwocC

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$$\underbrace{\text{There is } s\in\mathbb{R} \text{ with } s^n = y \text{ for } y>0 \text{ and } n\geq 2}$$

Proof:

consider set $A = \{x \in \mathbb{R} : x^n < y\}$

$0 \in A \Rightarrow A \neq \emptyset$

$\forall x \in A: \ x < y \Rightarrow y$ upper bound

$\Rightarrow$ $s := \sup A$ exists

Claim : $s^n = y$

We show this by eliminating the possibilities $s^n < y$, $s^n > y$

Assume $s^n < y$

We will work with expression $(s+\varepsilon)^n$

$$(s+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n + \varepsilon \cdot \sum_{k=1}^{n} \binom{n}{k} s^{n-k} \varepsilon^{k-1}$$

we will make sure that $0 < \varepsilon < 1$

$$\Rightarrow \quad (s+\varepsilon)^n < s^n + \varepsilon \underbrace{\sum_{k=1}^{n} \binom{n}{k} s^{n-k}}_{B:=}$$

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Now choose ε such that

$$0 < \varepsilon < \min\left(1, \frac{y - s^n}{B}\right)$$

$$\Rightarrow (s+\varepsilon)^n < s^n + \varepsilon \cdot B < s^n + \frac{y - s^n}{B} \cdot B = y$$

$$\Rightarrow s + \varepsilon \in A \quad \lightning \text{ contradicts } s = \sup A$$

Assume $s^n > y$

We will work with expression $(s-\varepsilon)^n$

$$(s-\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (-\varepsilon)^k$$

$$= s^n + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} (-\varepsilon)^k$$

$$= s^n + \sum_{\substack{k=1 \\ k \equiv 0 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$- \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$> s^n - \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n - \varepsilon \cdot \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^{k-1}$$

Again, we make sure $0 < \varepsilon < 1$

$$\Rightarrow (s-\varepsilon)^n > s^n - \varepsilon \underbrace{\sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k}}_{C :=}$$

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Now we choose ε such that

$$0 < \varepsilon < \min\left(1, \frac{s^n - y}{C}\right)$$

$$\Rightarrow (s-\varepsilon)^n > s^n - \varepsilon \cdot C > s^n - \frac{s^n - y}{C} \cdot C = y$$

$\Rightarrow$ $s - \varepsilon$ is upper bound of A

$\lightning$ contradicts $s = \sup A$

So $s^n = y$

$\blacksquare$

Bibliography

F. Lárusson. *Lectures on Real Analysis*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2012. ISBN 9781107026780. URL https://books.google.com/books?id=koj-IrXXwocC.