Completeness

COMPLETENESS and related properties[1] are the topic in this section.

Consider the function $f : \mathbb{Q} \to \mathbb{Q}$ defined as follows:

$$f(x) = \begin{cases} -1 & : x^2 < 2 \\ 1 & : \text{otherwise} \end{cases}$$

Even though $\forall x \in \mathbb{Q} : f'(x) = 0$ the function f is not constant. Furthermore f is continuous in $\mathbb{Q}$ and $f(0) = -1 < 0$ and $f(2) = 1 > 0$ but there is no $c \in \mathbb{Q}$ for which $f(c) = 0$, so the *Intermediate Value Property* doesn't hold[2].

Clearly $\mathbb{R}$ has an additional property which distinguishes it from $\mathbb{Q}$. This property cannot be deduced from the ordered field axioms[3] because those are shared by $\mathbb{Q}$ and $\mathbb{R}$ and we would be able to deduce it for $\mathbb{Q}$ too. It needs to be an additional property. The **Dedekind Completeness Property** is most commonly used as this additional property. We want to explore in this section how Dedekind Completeness relates to other properties also tied to what makes $\mathbb{R}$ different from $\mathbb{Q}$.

The properties we consider are[4]:

Dedekink Completeness Property **DDC**: Every non-empty real set bounded from above has a least upper bound.

Cut Property **CP**: Let A and B be two non-empty subsets of $\mathbb{R}$ with $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$ such that $\forall a \in A$ and $b \in B : a < b$. Then there exists a cutpoint $c \in \mathbb{R}$ such that $\forall a \in A$ and $b \in B : a \leq c \leq b$.

Archimedean Property **AP**: $\forall x \in \mathbb{R} : \exists n \in \mathbb{N}$ with $n > x$.

Nested Interval Property **NIP**: Given sequence of non-empty intervals $I_n, n \in \mathbb{N}$ with $I_{n+1} \subseteq I_n$, then $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Monotone Convergence Property **MC**: A bounded monotone sequence converges.

[1] Exercise 2.6.7 on page 71 from Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

[2] The Ancient Greeks already discovered that $\sqrt{2} \notin \mathbb{Q}$.

[3] We mean here the axioms of Addition and Multiplication (Commutativity, Associativity, etc) and Order axioms (Trichotemy, Transitivity, etc). See `http://homepages.math.uic.edu/~kauffman/axioms1.pdf`

[4] For a more detailed view on this topic and counterexamples of ordered fields without some of these properties see J. Propp. Real Analysis in Reverse. *ArXiv e-prints*, April 2012. URL `https://arxiv.org/abs/1204.4483`.

Bolzano-Weierstrass Property **BW**: A bounded sequence has a convergent subsequence.

Cauchy Criterion **CC**: A sequence converges if and only if it is a Cauchy sequence.

Ratio Test **RT**: If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges[5].

Intermediate Value Property **IV**: Given is a continuous function $f : [a, b] \to \mathbb{R}$ with $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in [a, b]$ with $f(c) = 0$.

[5] The *Ratio Test* and the *Intermediate Value Property* feel like higher level properties that use infinite series and continuos functions. We will see in the following theorems how they relate to the other properties.

Theorem 1.1. *$DDC \Leftrightarrow CP$*

Proof. ($\Rightarrow$) We have A and B two non-empty subsets of $\mathbb{R}$ with $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$ such that $\forall a \in A$ and $b \in B : a < b$. B is non-empty, so there exists $b \in B$. This b is an upper bound of A, so A is bound from above. By the Dedekind Completeness Property DDC there exists a least upper bound c. We claim that c is the desired cutpoint. Since c is the least upper bound we already have $A \leq c$. Assume $\exists b' \in B$ with $b' < c$. But b' is an upper bound of A (since $A < B$) which means $c \leq b'$ because c is the least upper bound. This is a contradiction, so $\forall b' \in B : b' \geq c$. It follows that $A \leq c \leq B$ and c is the cutpoint.
($\Leftarrow$) We are given a non-empty set $A \subset \mathbb{R}$ bound from above, so there exists $b \in \mathbb{R} : A \leq b$. We define B be the set of upper bounds of A and let $A' = \mathbb{R} \setminus B$. Both A' and B are non-empty, $A' < B$ and $A' \cup B = \mathbb{R}$[6]. By the Cut Property CP there exists a cutpoint c with $A' \leq c \leq B$. We claim that c is the least upper bound of A. Assume there exists $a \in A$ with $c < a$. Then for $c' = \frac{c+a}{2}$ we have $c < c' < a$. This implies that $c' \in B$ so c' is an upper bound of A which contradicts with $c' < a$. We therefore have $\forall a \in A : a \leq c$ and c is an upper bound of A. Now assume there exists another upper bound d with $d < c$. But then $d \in A'$ which contradicts the definition of A' and B. So for all d upper bound of A we have $d \geq c$. This makes c the least upper bound of A. □

[6] The set A is bounded from above so B is non-empty. If $A = \{a\}$ then A' is non-empty (for example $(a-1) \in A'$). If $|A| > 1$ then one of the elements in A cannot be an upper bound of A which also implies A' is non-empty. By definition $A' \cup B = \mathbb{R}$. Assume there exists $a' \in A'$ and $b' \in B$ such that $a' \geq b'$. This would make a' an upper bound of A, so $a' \in B$, a contradiction. It follows that $A' < B$.

Theorem 1.2. *$DDC \Leftrightarrow NIP + AP$*[7]

Proof. ($\Rightarrow$) We have nested intervals $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$. It follows that for all $n \in \mathbb{N}$ we have $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$. Assume there exists $i, j \in \mathbb{N}$ such that $b_i < a_j$. We have three cases:

- $i = j$: then $a_i \leq b_i$ for interval I_i contradicting $b_i < a_j$.
- $i < j$: then $b_i \geq b_j$ which yields the inequality chain $b_j \leq b_i < a_j$, contradicting $a_j \leq b_j$ for interval I_j.

[7] The Nested Intervals Property NIP is not enough to achieve Dedekind Completeness DDC. For examples of fields that are not Archimedean see J. Propp. Real Analysis in Reverse. *ArXiv e-prints*, April 2012. URL `https://arxiv.org/abs/1204.4483`. This theorem only shows that if the Archimedean Property AP also holds then we can get back from NIP to DDC.

- $i > j$: then $a_i \geq a_j$ which yields the inequality chain $b_i < a_j \leq a_i$, contradicting $a_i \leq b_i$ for interval I_i.

This means that for all $i, j \in \mathbb{N}$ we have $a_j \leq b_i$. In other words, the b_n are upper bounds for the set $A = \{a_n : n \in \mathbb{N}\}$.

The set A is bound from above and non-empty, so according to DDC there exists a least upper bound c. Since it is an upper bound we already have $\forall n \in \mathbb{N} : a_n \leq c$. Since c is the least upper bound and all b_n are upper bounds we also have $c \leq b_n$. It follows that $\forall n \in \mathbb{N} : c \in I_n$ or $c \in \cap_{n \in \mathbb{N}} I_n$. This proves $DDC \Rightarrow NIP$.

Assume there exists $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : n \leq x$. This means that $\mathbb{N}$ is bound from above. Let c be the least upper bound for $\mathbb{N}$. We have

$$\forall n \in \mathbb{N} : n + 1 \in \mathbb{N} \Rightarrow n + 1 \leq c \Rightarrow n \leq c - 1$$

$c - 1$ is an upper bound, c is the least upper bound so $c \leq c - 1$, a contradiction. This proves $DDC \Rightarrow AP$.

($\Leftarrow$) Consider the non-empty set $S \subseteq \mathbb{R}$ bounded from above by $b_0 \in \mathbb{R}$.

We want to apply NIP, so we define nested intervals around the upper bounds of S.

Proof Part 1.2.1. S is non-empty, so there exists $a_0 \in S$. Define $I_0 = [a_0, b_0]$. The strategy now is to halve the interval and narrow it down but remain with the right endpoint of each interval "on top of" S and with the left endpoint in S.

Consider $m = \frac{a_0+b_0}{2}$. If $[m, b_0] \cap S = \emptyset$ then let $a_1 = a_0$ and $b_1 = m$. If on the other hand $\exists s \in [m, b_0] \cap S$ then let $a_1 = s$ and $b_1 = b_0$. Define $I_1 = [a_1, b_1]$. Repeat this process to define all $I_n, n \in \mathbb{N}$.

The intervals I_n have the following properties:

P1 : $I_{n+1} \subseteq I_n$. This is visible from the definition of I_{n+1}. Its endpoints are either endpoints of I_n or are points from inside I_n.

P2 : $\forall n \in \mathbb{N} : b_n$ upper bound of S. We show this by induction on n. By choice b_0 is an upper bound. Now assume that b_n is an upper bound. If $b_{n+1} = b_n$ then it is an upper bound. If $b_{n+1} = \frac{a_n+b_n}{2}$ then because $S \cap [b_{n+1}, b_n] = \emptyset$ and it also follows that b_{n+1} is an upper bound [8].

P3 : $\forall n \in \mathbb{N}$: I_n non-empty. This also follows by induction and by the field axioms of $\mathbb{R}$.

P4 : $\forall n \in \mathbb{N} : a_n \in S$. This follows by induction and definition of left endpoints.

P5 : $\forall n \in \mathbb{N} : |I_n| \leq \frac{b_0-a_0}{2^n}$. [9]

[8] Assume b_{n+1} is not an upper bound of S, so there exists $s' \in S$ with $s' > b_{n+1}$. But by induction b_n is an upper bound, which means $b_{n+1} < s' \leq b_n$, so $s' \in [b_{n+1}, b_n]$, which contradicts $S \cap [b_{n+1}, b_n] = \emptyset$.

[9] We show this by induction on n. Base case $n = 0$ holds by definition of I_0. Assume $|I_n| \leq \frac{b_0-a_0}{2^n}$. For I_{n+1} we observe that its length is either half that of I_n or less than half when $[\frac{a_n+b_n}{2}, b_n] \cap S \neq \emptyset$

$P1$ and $P3$ satisfy the requirements of NIP, so we know $\alpha \in \cap_{n\in\mathbb{N}} I_n$ exists.

We want to show that $\alpha = supS$.

Proof Part 1.2.2. Assume α is not an upper bound of S. Then there exists $s \in S$ with $s > \alpha$. Let $\epsilon = s - \alpha > 0$. Using the Archimedean property we choose $m \in \mathbb{N}$ such that $I_m = [a_m, b_m]$ with $|I_m| < \epsilon$ [10]. Then $\alpha \in I_m$, but $s \notin I_m$ and furthermore $b_m < s$. This is a contradiction to property $P2$, so α is an upper bound of S.

[10] We use property $P5$. From $|I_m| \le \frac{b_0 - a_0}{2^m} < \epsilon$, we get $m > log_2(\frac{b_0 - a_0}{\epsilon})$.

Now assume α is not the smallest upper bound of S. Then there exists an upper bound β of S with $\beta < \alpha$. Let $\epsilon = \alpha - \beta > 0$. Again we choose $m \in \mathbb{N}$ such that $I_m = [a_m, b_m]$ with $|I_m| < \epsilon$. That pushes a_m between β and α: $\beta < a_m \le \alpha$. But according to property $P4$, $a_m \in S$, so $\beta < a_m$ contradicts the fact that β is an upper bound of S. So α is the smallest upper bound of S: $\alpha = supS$. This proves $NIP + AP \Rightarrow DDC$

□

Theorem 1.3. $DDC \Leftrightarrow MC$

Proof. ($\Rightarrow$) Given is a monotone increasing sequence (a_n) bound from above. We define $A = \{a_n : n \in \mathbb{N}\}$, a set that is bound from above. From DDC it follows that least upper bound c of A exists. We want to show that $\lim_{n\to\infty} a_n = c$. For all $\epsilon > 0$ we have $c - \epsilon < c$, so $c - \epsilon$ cannot be an upper bound of A (c is the least upper bound). That means that there exists $n_0 \in \mathbb{N}$ with $a_{n_0} > c - \epsilon$. Since the sequence is monotone increasing, we have

$$\forall n \ge n_0 : a_n \ge a_{n_0} > c - \epsilon \Rightarrow |c - a_n| < \epsilon$$

which proves $a_n \to c$.
($\Leftarrow$) We first want to show $MC \Rightarrow AP$. Given MC assume that AP doesn't hold, so there exists $x \in \mathbb{R}$ bigger than any natural number. This means x is an upper bound for the sequence $a_n = n$, a monotone increasing sequence. From MC it then follows that a_n converges to a limit c. The sequence $b_n = n + 1$ is a_n shifted to the left, so it is also convergent with the same limit c. Taking the limit on the sequence equation $b_n = a_n + 1$ we get $c = c + 1$, a contradiction. So $MC \Rightarrow AP$.

To show that $MC \Rightarrow DDC$ we are given non-empty set S with $a_0 \in S$ bound from above by $b_0 \in \mathbb{R}$. We define the same nested intervals as in the Proof Part 1.2.1 of the proof of Theorem 1.2.

The same properties P_1 to P_5 for I_n as stated in Proof Part 1.2.1 hold. The sequence (a_n) is in S and monotone increasing and the sequence (b_n) is made of upper bounds of S and is monotone decreasing. (a_n) is bound from above and monotone so according to MC it converges to a limit α.

We want to show that $\alpha = supS$. We will use the exact same argument as in the Proof Part 1.2.2 of the proof of Theorem 1.2[11]. This proves $MC \Rightarrow DDC$ □

Theorem 1.4. $DDC \Leftrightarrow BW + AP$[12]

Proof. $(\Rightarrow)$ We have already seen $DDC \Rightarrow AP$ (Theorem 1.2).

Proof Part 1.4.1. To prove $DDC \Rightarrow BW$ we are given a bounded sequence (s_n):

$$\exists a_0, b_0 \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N} : a_0 \leq s_n \leq b_0$$

We define interval $I_0 = [a_0, b_0]$ and divide it in half at $c = \frac{a_0+b_0}{2}$. At least one of the two intervals $[a_0, c]$, $[c, b_0]$ has an infinite number of elements of the sequence s_n[13]. Define I_1 to be either $[a_0, c]$ or $[c, b_0]$ with an infinite number of elements of s_n. We repeat this process recursively, defining I_m to be one of the halves of I_{m-1} that has an infinite number of elements of (s_n). We get a sequence of nested intervals (I_m) of decreasing length $|I_m| = \frac{a_0+b_0}{2^m}$.

We define $f : \mathbb{N} \to \mathbb{N}$ recursively as

$$\begin{cases} f(1) & = 1 \\ f(n) & = min\{i > f(n-1) : s_i \in I_{n-1}\} \end{cases}$$

The set $\{i > f(n-1) : s_i \in I_{n-1}\}$ is a non-empty, infinite subset[14] of $\mathbb{N}$, so its minimum exists and f is well defined and by definition strictly monotone increasing. We define subsequence (s'_n) as $s'_n = s_{f(n)}$, well defined because f is strictly monotone increasing.

Proof Part 1.4.2. From DDC we know that NIP holds so $\alpha \in \cap_{m\in\mathbb{N}} I_m$ exists. We claim that $s'_n \to \alpha$.

Because of AP we have for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|I_{n_0}| < \epsilon$. We have $\alpha \in I_{n_0}$ and for all $n > f^{-1}(n_0) : s'_n \in I_{n_0}$. This means for all $n > f^{-1}(n_0) : |s'_n - \alpha| < \epsilon$ and (s'_n) is a subsequence of (s_n) that converges to α.

$(\Leftarrow)$

Proof Part 1.4.3. We are going to prove this direction by going through NIP. Given nested non-empty intervals $I_{n+1} \subseteq I_n$ we define sequence (s_n) by choosing an arbitrary element from each I_n and setting it to be s_n. According to BW there exists a subsequence (s'_n) of (s_n) that converges $s'_n \to c$. We claim that $c \in \cap_{n\in\mathbb{N}} I_n$.

Proof Part 1.4.4. Assume $c \notin \cap_{n\in\mathbb{N}} I_n$. Then there must exist $n_0 \in \mathbb{N}$ such that $c \notin I_{n_0} = [a_{n_0}, b_{n_0}]$. Either $c < a_{n_0}$ or $c > b_{n_0}$. Let's consider $c < a_{n_0}$ (the other case is very similar). $\epsilon = \frac{a_{n_0}-c}{2} > 0$. We have $s'_n \to c$, so there exists n_1 such that $\forall n > n_1 : |s'_n - c| < \epsilon$. So for

[11] The only difference in the two proofs is that in this proof MC ensures the existence of α and in the previous proof it was NIP.

[12] Once again Bolzano-Weierstrass BW is not enough to get back to Dedekind Completeness DDC. We need the field to be Archimedean AP.

[13] Otherwise (s_n) would not be an infinite sequence.

[14] By definition of I_{n-1} there are an infinite number of elements s_i in I_{n-1}, so there are an infinite number of indices i in $\{i > f(n-1) : s_i \in I_{n-1}\}$. Also any non-empty subset of $\mathbb{N}$ has a smallest element.

$\forall n > max(n_0, n_1) : s'_n < c + \epsilon < a_{n_0}$. But (s'_n) is a subsequence of (s_n) so there must exist $m \in \mathbb{N}$ with $f^{-1}(m) > max(n_0, n_1)$. We have $s'_m = s_{f^{-1}(m)} \in I_{f^{-1}(m)}$. So $s'_m \in I_{f^{-1}(m)} \subseteq I_{n_0}$ and $s'_m < a_{n_0}$ which is a contradiction. This means $c \in \cap_{n \in \mathbb{N}} I_n$ and $BW \Rightarrow NIP$ which together with AP gets us to DDC according to Theorem 1.2.

□

Theorem 1.5. $DDC \Leftrightarrow CC + AP$[15]

Proof. ($\Rightarrow$) We have already seen $DDC \Rightarrow AP$ (Theorem 1.2). To prove $DDC \Rightarrow CC$ we are given a Cauchy sequence (a_n). We first show that (a_n) is bounded. From the definition of a Cauchy sequence[16] we get for $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that $\forall m \geq N : |a_n - a_N| < 1 \Rightarrow |a_n| < 1 + |a_N|$. Define $M = max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$ and we have $\forall n \in \mathbb{N} : |a_n| < M$.

The Cauchy sequence (a_n) is bounded so using $DDC \Rightarrow BW$ from Theorem 1.4 we know there is a subsequence of (a_n) that converges. Let $f : \mathbb{N} \to \mathbb{N}$ be the strictly monotone increasing function that defines the converging subsequence $a'_n = a_{f(n)}$ and let $\lim_{n\to\infty} a'_n = c$.

For all $\epsilon > 0$ we have:

$$\exists n_1 \in \mathbb{N} \text{ such that } \forall n \geq n_1 : |a_n - a_{n_1}| < \frac{\epsilon}{2}$$

and then

$$\exists n_2 \geq f^{-1}(n_1) \text{ such that } \forall n \geq n_2 : |a'_n - c| < \frac{\epsilon}{2}$$

So

$$\begin{aligned} \forall n \geq n_2 : |a_n - c| &= |a_n - a'_{n_2} + a'_{n_2} - c| \leq |a_n - a'_{n_2}| + |a'_{n_2} - c| \\ &= |a_n - a_{f(n_2)}| + |a'_{n_2} - c| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

It means (a_n) converges to c and $DDC \Rightarrow CC + AP$.

($\Leftarrow$) We will show that $CC + AP \Rightarrow BW$. We are given a bounded sequence (s_n) and we use the same subsequence construction as in the Proof Part 1.4.1 of Theorem 1.4. We claim that the so constructed subsequence (s'_n) is a Cauchy sequence. Indeed for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|I_N| < \epsilon$ (again we need AP here). We then have:

$$\forall m, n \geq N : s'_n, s'_m \in I_N \Rightarrow |s'_n - s'_m| \leq |I_N| < \epsilon$$

So (s'_n) is a Cauchy sequence and by CC it converges which means that (s_n) has a convergent subsequence. □

Finish up.

[15] As seen before with NIP and BW the Cauchy Criterion CC is not enough to get back to Dedekind Completeness DDC. We need the field to be Archimedean AP.

[16] A sequence (a_n) is a Cauchy sequence if $\forall \epsilon > 0 : \exists N \in \mathbb{N}$ such that $\forall m, n \geq N : |a_m - a_n| < \epsilon$.

Bibliography

Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

J. Propp. Real Analysis in Reverse. *ArXiv e-prints*, April 2012. URL https://arxiv.org/abs/1204.4483.