Schröder-Bernstein Theorem

BIJECTIONS from one-to-one functions are the topic[1] in this note. The problem statement is known as the Schröder-Bernstein Theorem.

[1] Exercise 1.5.11 on page 32 from Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

Problem

Let $f : X \to Y$ and $g : Y \to X$ be one-to-one functions. Then there exists a bijection $h : X \to Y$.

The given functions are one-to-one, so for subsets $f(X)$ and $g(Y)$ they are already bijections. This leads to the idea of partitioning X and Y such that we can compose a bijection h piece-wise from f and g^{-1} using the partitions. In particular given a subset $A \subseteq X$, we consider the sets A, $X \setminus A$, $f(A)$, $Y \setminus f(A)$ and $g(Y \setminus f(A))$. We want subsets $A \subseteq X$, such that $A \cap g(Y \setminus f(A)) = \emptyset$, as shown in figure 1.2. Let's define this as property P:

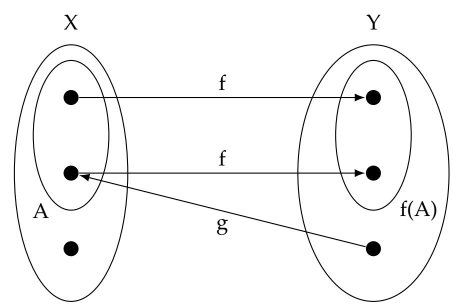

Figure 1.1: A violates $P(A)$

$$\forall A \subseteq X : P(A) \Leftrightarrow A \cap g(Y \setminus f(A)) = \emptyset$$

If we have a subset $A \subseteq X$ that satisfies $P(A)$, then we can define the bijection h:

$$h(x) = \begin{cases} f(x) & : x \in A \\ g^{-1}(x) & : x \in g(Y \setminus f(A)) \end{cases}$$

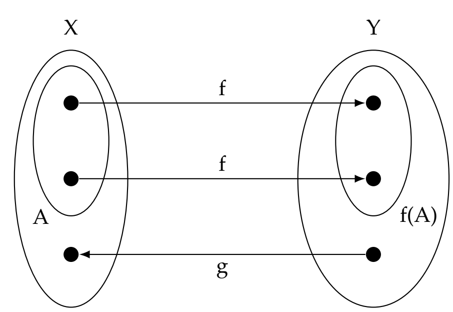

Figure 1.2: A satisfies $P(A)$

The domain of h is $A \cup g(Y \setminus f(A))$, which is not necessarily equal to X, so we are not done yet. Our goal therefore is to find a subset $A \subseteq X$ that satisfies $P(A)$ and for which $A \cup g(Y \setminus f(A)) = X$.

Let

$$\Lambda = \{A \subseteq X : P(A)\}$$

be the set of all subsets of X that satisfy property P and let $\bar{A}$ be the union of all such subsets

$$\bar{A} = \bigcup_{A \in \Lambda} A$$

Lemma 1.1. *$\bar{A}$ is the biggest subset of X that satisfies P.*

Proof. First we show that $\bar{A}$ satisfies P. Assume

$$\exists y \in Y \setminus f(\bar{A}) \text{ with } g(y) \in \bar{A}$$

Then there exists a set $A \in \Lambda$ with $g(y) \in A$ [2]. $A \subseteq \bar{A}$, so $f(A) \subseteq f(\bar{A})$. Therefore $Y \setminus f(\bar{A}) \subseteq Y \setminus f(A)$, so $y \in Y \setminus f(A)$. But this contradicts A satisfying property P, so no such y exists. It follows that $\bar{A}$ satisfies P too.

[2] Because $\bar{A} = \bigcup_{A \in \Lambda}$.

Assume there is a set A' that satisfies P and that is bigger than $\bar{A}$, so $\bar{A} \subseteq A'$. But $A' \in \Lambda$ and $\bar{A} = \bigcup_{A \in \Lambda}$, so $A' \subseteq \bar{A}$. That means $A' = \bar{A}$. □

With $\bar{A}$ we can define the partitions $X = \bar{A} \oplus (X \setminus \bar{A})$ and $Y = f(\bar{A}) \oplus (Y \setminus f(\bar{A}))$.

Lemma 1.2.

$$g(Y \setminus f(\bar{A})) = X \setminus \bar{A}$$

Proof. Because $\bar{A}$ satisfies P, we already know that

$$g(Y \setminus f(\bar{A})) \subseteq X \setminus \bar{A}$$

Now assume

$$\exists x \in X \setminus \bar{A} \text{ such that } \forall y \in Y \setminus f(\bar{A}) : g(y) \neq x$$

But then $\bar{A} \cup \{x\}$ satisfies P [3] and is bigger than $\bar{A}$. This contradicts lemma 1.1. So no such x exists and the lemma is proven. □

[3] We have

$$Y \setminus f(\bar{A} \cup \{x\}) \subseteq Y \setminus f(\bar{A})$$

so

$$\forall y \in Y \setminus f(\bar{A} \cup \{x\}) : g(y) \notin \bar{A} \cup \{x\}$$

We can now define the bjection $h : X \to Y$ with

$$h(x) = \begin{cases} f(x) & : x \in \bar{A} \\ g^{-1}(x) & : x \in X \setminus \bar{A} \end{cases}$$

which solves the problem in this section. [4]

[4] The solution uses a nifty proof strategy: maximize a mathematical structure so that its "complement" has no choice but to satisfy a certain property, ie not satisfying the property would contradict the maximality.

Bibliography

Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.